

MATRIX COMPLETION AND TENSOR RANK

HARM DERKSEN

ABSTRACT. In this paper, we show that the low rank matrix completion problem can be reduced to the problem of finding the rank of a certain tensor.

1. INTRODUCTION

Suppose we are given a rectangular array which has only been partially filled out with entries in \mathbb{R} (or some other field). The low rank matrix completion problem asks to fill out the remaining entries such that the resulting matrix has minimal rank. For example, the array

$$\begin{pmatrix} 2 & \cdot & \cdot \\ 2 & 3 & \cdot \\ \cdot & 6 & 2 \end{pmatrix}$$

can be completed to a rank 1 matrix

$$\begin{pmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 4 & 6 & 2 \end{pmatrix}$$

but clearly cannot be completed to a rank 0 matrix. For a survey on matrix completion problems, see [14, 12]. The low rank matrix completion problem has various applications, such as statistics, computer vision, signal processing and control. The low rank matrix completion problem is known to be NP-hard (see [3]). For the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ one can solve the matrix completion problem using convex relaxation: under certain assumptions minimizing the nuclear norm of the matrix (the sum of the singular values) one also obtains the minimal rank completion. See for example [18, 4, 5, 15]. The rank of a tensor was defined in [11]. Using this notion, matrix completion problems can be generalized to higher order tensors (see [7]).

Suppose that \mathbb{F} is a field, and $V^{(i)} = \mathbb{F}^{m_i}$ for $i = 1, 2, \dots, d$. The tensor product space

$$V = V^{(1)} \otimes V^{(2)} \otimes \dots \otimes V^{(d)}$$

can be viewed as a d -dimensional array of size $m_1 \times m_2 \times \dots \times m_d$. A *pure* tensor is an element of V of the form

$$v^{(1)} \otimes v^{(2)} \otimes \dots \otimes v^{(d)},$$

where $v^{(i)} \in V^{(i)}$ for all i . The rank of a tensor $T \in V$ is the smallest nonnegative integer r such that T can be written as a sum of r pure tensors:

$$T = \sum_{i=1}^r v_i^{(1)} \otimes v_i^{(2)} \otimes \dots \otimes v_i^{(d)}$$

The author was partially supported by NSF grant DMS 0901298.

where $v_j^{(i)} \in V^{(i)}$ for all i and j . We denote the tensor rank of T by $\text{rank}(T)$. If $A = (a_{i,j})$ is an $n \times m$ matrix, then we can identify A with the tensor

$$\sum_{i=1}^n \sum_{j=1}^m a_{i,j} (e_i \otimes e_j),$$

in $\mathbb{F}^n \otimes \mathbb{F}^m$, where e_i denotes the i -th basis vector. The tensor rank in this case is exactly the same as the rank of the matrix. For $d \geq 3$ it is often difficult to find the rank of a given tensor. Finding the rank of a tensor is NP-complete if the field \mathbb{F} is finite ([9, 10]) and NP-hard if the field is \mathbb{Q} ([9, 10]), or if it contains \mathbb{Q} ([13]). Approximation of a tensor by another tensors of low rank is known as the PARAFAC ([8]) or CANDECOMP ([2]) model. There are various algorithms for finding low-rank approximations. See [16, 19] for a discussion. The problems of finding the tensor rank of a tensor, and to approximate tensors with tensors of low rank has many applications, such as the complexity of matrix multiplication, fluorescence spectroscopy, statistics, psychometrics, geophysics and magnetic resonance imaging. In general, the rank of higher tensors more ill-behaved than the rank of a matrix (see [6]).

In the next section we will show that the low rank matrix completion problem can be reduced to finding the rank of a certain tensor. Lek-Heng Lim pointed out to the author, that because low rank matrix completion is NP-hard (see [3]), this gives another proof that determining the rank of a tensor is NP-hard.

2. THE MAIN RESULT

Suppose that \mathbb{F} is a field, and $A = (a_{i,j})$ is an $n \times m$ array for which some of the entries are given, and the entries in positions $(i_1, j_1), \dots, (i_s, j_s)$ are undetermined. Define $e_{i,j}$ as the matrix that has a 1 in position (i, j) and 0 everywhere else. We can reformulate the low rank matrix completion problem as follows. Define $a_{i_k, j_k} = 0$ for $k = 1, 2, \dots, s$ and find a matrix B of the form

$$B = A + \sum_{k=1}^s \lambda_k e_{i_k, j_k}, \quad \lambda_1, \dots, \lambda_s \in \mathbb{F}.$$

which has minimal rank.

We can view matrices as second order tensors. So we identify $e_{i,j}$ with $e_i \otimes e_j \in \mathbb{F}^n \otimes \mathbb{F}^m$ and we have

$$A = \sum_{i=1}^n \sum_{j=1}^m a_{i,j} (e_i \otimes e_j).$$

Define $u_k = e_{i_k}$ and $v_k = e_{j_k}$. The matrix completion problem asks to find $\lambda_1, \dots, \lambda_s \in \mathbb{F}$ such that the tensor

$$B = A + \sum_{k=1}^s \lambda_k (u_k \otimes v_k) = \sum_{i=1}^n \sum_{j=1}^m a_{i,j} (e_i \otimes e_j) + \sum_{k=1}^s \lambda_k (u_k \otimes v_k)$$

has minimal rank.

Let $U \subseteq \mathbb{F}^n \otimes \mathbb{F}^m$ be the vector space spanned by $u_1 \otimes v_1, \dots, u_s \otimes v_s$.

Definition 1. We define

$$\text{rank}(A, U) = \min\{\text{rank}(B) \mid B \in A + U\}.$$

So the matrix completion asks for the value of $\text{rank}(A, U)$ and to find a matrix $B \in A + U$ for which $\text{rank}(A, U) = \text{rank}(B)$.

We define a third order tensor $\hat{A} \in \mathbb{F}^n \otimes \mathbb{F}^m \otimes \mathbb{F}^{s+1}$ by

$$(1) \quad \hat{A} = A \otimes e_{s+1} + \sum_{k=1}^s u_k \otimes v_k \otimes e_k = \sum_{i=1}^n \sum_{j=1}^m a_{i,j} (e_i \otimes e_j \otimes e_{s+1}) + \sum_{k=1}^s u_k \otimes v_k \otimes e_k.$$

Our main result is:

Theorem 2. *We have*

$$\text{rank}(A, U) + s = \text{rank}(\hat{A}).$$

The theorem is effective: given an explicit decomposition of \hat{A} as a sum of $l = \text{rank}(\hat{A})$ pure tensors, the proof shows that one easily can construct a matrix B in $A + U$ such that $\text{rank}(B) = l - s = \text{rank}(A, U)$.

Proof of Theorem 2. Let $r = \text{rank}(A, U)$ and $l = \text{rank}(\hat{A})$. By definition, there exists a matrix $B \in A + U$ with rank r . We can write

$$B = A + \sum_{k=1}^s \lambda_k (u_k \otimes v_k).$$

Since B has rank r , we also can write

$$B = \sum_{i=1}^r x_i \otimes y_i$$

for some $x_1, \dots, x_r \in \mathbb{F}^n$ and $y_1, \dots, y_r \in \mathbb{F}^m$. We have

$$\begin{aligned} \hat{A} &= A \otimes e_{s+1} + \sum_{k=1}^s u_k \otimes v_k \otimes e_k = (B - \sum_{k=1}^s \lambda_k u_k \otimes v_k) \otimes e_{s+1} + \sum_{k=1}^s u_k \otimes v_k \otimes e_k = \\ &= B \otimes e_{s+1} + \sum_{k=1}^s u_k \otimes v_k \otimes (e_k - \lambda_k e_{s+1}) = \sum_{i=1}^r x_i \otimes y_i + \sum_{k=1}^s u_k \otimes v_k \otimes (e_k - \lambda_k e_{s+1}). \end{aligned}$$

We have written \hat{A} as a sum of $r + s$ pure tensors. So $l = \text{rank}(\hat{A}) \leq r + s$.

Conversely, we can write

$$\hat{A} = \sum_{j=1}^l a_j \otimes b_j \otimes c_j$$

For every i , choose a linear function $h_i : \mathbb{F}^n \otimes \mathbb{F}^m \rightarrow \mathbb{F}$ such that $h_i(u_i \otimes v_i) = 1$ and $h_i(u_j \otimes v_j) = 0$ if $j \neq i$. Applying $h_i \otimes \text{id} : \mathbb{F}^n \otimes \mathbb{F}^m \otimes \mathbb{F}^{s+1} \rightarrow \mathbb{F}^{s+1}$ to the tensor \hat{A} gives

$$\sum_{j=1}^l h_i(a_j \otimes b_j) c_j = (h_i \otimes \text{id})(\hat{A}) = e_i + h_i(A) e_{s+1}.$$

This shows that e_i lies in the span of c_1, \dots, c_l, e_{s+1} for all i . So c_1, \dots, c_l, e_{s+1} span the vector space \mathbb{F}^{s+1} . After rearranging c_1, \dots, c_l , we may assume that $c_1, c_2, \dots, c_s, e_{s+1}$ is a

basis of \mathbb{F}^{s+1} . There exists a linear function $f : \mathbb{F}^{s+1} \rightarrow \mathbb{F}$ such that $f(c_1) = \dots = f(c_s) = 0$ and $f(e_{s+1}) = 1$. We apply $\text{id} \otimes \text{id} \otimes f : \mathbb{F}^n \otimes \mathbb{F}^m \otimes \mathbb{F}^{s+1} \rightarrow \mathbb{F}^n \otimes \mathbb{F}^m$ to \hat{A} :

$$A + \sum_{i=1}^s f(e_i) u_i \otimes v_i = (\text{id} \otimes \text{id} \otimes f)(\hat{A}) = \sum_{j=1}^l f(c_j) a_j \otimes b_j = \sum_{j=s+1}^l f(c_j) a_j \otimes b_j.$$

The tensor above is a matrix in $A + U$ which has rank at most $l - s$, and by definition of $r = \text{rank}(A, U)$ it has rank at least r . It follows that $r \leq l - s$. We have already proven that $r + s \geq l$, so we conclude that $r + s = l$. \square

Remark 3. In the proof we have not used that u_k and v_k are basis vectors. So Theorem 2 is true whenever U is the span of linearly independent pure tensors

$$u_1 \otimes v_1, \dots, u_s \otimes v_s$$

and \hat{A} is given by (1).

Remark 4. Define

$$t_i \in (\mathbb{F}^2)^{\otimes s} = \underbrace{\mathbb{F}^2 \otimes \dots \otimes \mathbb{F}^2}_s$$

by

$$t_i = e_1 \otimes \dots \otimes e_1 \otimes e_2 \otimes e_1 \otimes \dots \otimes e_1$$

where e_2 appears in the i -th position if $i \leq s$, and

$$t_{s+1} = e_1 \otimes e_1 \otimes \dots \otimes e_1.$$

We define a tensor $\tilde{A} \in \mathbb{F}^n \otimes \mathbb{F}^m \otimes (\mathbb{F}^2)^{\otimes s}$ by

$$\tilde{A} = u_1 \otimes v_1 \otimes t_1 + \dots + u_s \otimes v_s \otimes t_s + A \otimes t_{s+1}.$$

A proof, similar to that of Theorem 2 shows that $\text{rank}(\tilde{A}) = s + \text{rank}(A, U)$. The ambient vector space of \tilde{A} is much larger than that of \hat{A} . But one can imagine that using the tensor \tilde{A} can be advantageous for proving lower bounds for $\text{rank}(A, U)$ because it is, in a way, more rigid.

Remark 5. Theorem 2 also easily generalizes to the higher order analog of matrix completion: *tensor completion*. Suppose that $V^{(1)}, \dots, V^{(d)}$ are finite dimensional \mathbb{F} -vector spaces and $V = V^{(1)} \otimes \dots \otimes V^{(d)}$. Assume that $U \subseteq V$ is a subspace spanned by s linearly independent pure tensors

$$u_i^{(1)} \otimes u_i^{(2)} \otimes \dots \otimes u_i^{(d)}, \quad i = 1, 2, \dots, s$$

and

$$A \in V^{(1)} \otimes \dots \otimes V^{(d)}$$

is a tensor. As before, we define

$$\text{rank}(A, U) = \min\{\text{rank}(B) \mid B \in A + U\}.$$

Define a $(d+1)$ -th order tensor by

$$\hat{A} = A \otimes e_{s+1} + \sum_{k=1}^s u_k^{(1)} \otimes \dots \otimes u_k^{(d)} \otimes e_k \in V^{(1)} \otimes \dots \otimes V^{(d)} \otimes \mathbb{F}^{s+1}.$$

Then we have

$$\text{rank}(A, U) + s = \text{rank}(\hat{A}).$$

Acknowledgment. The author thanks Lek-Heng Lim for some useful suggestions.

REFERENCES

- [1] E. Acar, B. Yener, *Unsupervised multiway data analysis: a literature study*, IEEE Trans. on Knowledge and Data Engin. **21** (2009), 6–20.
- [2] J. D. Carroll and J. Chang, *Analysis of individual differences in multidimensional scaling via an n -way generalization of “Eckart-Young” decomposition*, Psychometrika **35** (1970), 283–319.
- [3] A. L. Chistov and Yu. Grigoriev, *Complexity of quantifier elimination in the theory of algebraically closed fields*, in: Proceedings of the 11th Symposium on Mathematical Foundations of Computer Science, Springer Verlag, 1984.
- [4] E.J. Candès and B. Recht, *Exact matrix completion via convex optimization*, Foundations of Computational Mathematics **9**, 2009, 717–772.
- [5] E. J. Candès and T. Tao, *The power of convex relaxation: Near-optimal matrix completion*, Preprint.
- [6] V. De Silva and L.-H. Lim, *Tensor rank and the ill-posedness of the best low-rank approximation problem*, SIAM J. Matrix Analysis Appl. **30** (2008), no. 3, 1084–1127.
- [7] S. Gandy, B. Recht and I. Yamada, *Tensor completion and low n -rank tensor recovery via convex optimization*, Inverse Problems **27** (2011), no. 2.
- [8] R. A. Harshman, *Foundations of the parafac procedure: models and conditions for an “explanatory” multimodal factor analysis*, UCLA working papers in Phonetics **16** (1970), 1–84.
- [9] J. Håstad, *Tensor rank is NP-complete*, J. Algorithms **11** (1990), no. 4, 644–654.
- [10] J. Håstad, *Tensor rank is NP-complete*, Automata, languages and programming (Stresa, 1989), Lecture Notes in Comput. Sci. **372**, Springer, Berlin, 1989, 451–460.
- [11] F. L. Hitchcock, *Multiple invariants and generalized rank of a p -way matrix or tensor*, J. Math. and Phys. **7** (1927), no. 1, 40–79.
- [12] M. Laurent, *Matrix completion problems*, in: The Encyclopedia of Optimization **III**, C. A. Floudas and P. M. Pardalos, eds., Kluwer, 2001.
- [13] C. Hillar and L.-H. Lim, *Most tensor problems are NP-hard*, Preprint.
- [14] C. R. Johnson, *Matrix completion problems: a survey*, in: Matrix Theory and Applications, AMS, Providence, RI, 1990, 171–198.
- [15] R. H. Keshavan, A. Montanari and S. Oh, *Matrix completion from a few entries*, IEEE Trans. on Information Theory **56** (2010), 2980–2998.
- [16] T. G. Kolda, B. W. Bader, *Tensor Decompositions and Applications*, SIAM Rev. **51** (3), 455–500.
- [17] B. Recht, *A simpler approach to matrix completion*, J. Mach. Learn. Res. **12** (2011), 3413–3430.
- [18] B. Recht, M. Fazel and P. Parillo, *Guaranteed minimum rank solutions of matrix equations via nuclear norm minimization*, SIAM Review **52** (2010), no. 3, 471–501.
- [19] G. Tomasi and R. Bro, *A comparison of algorithms for fitting the parafac model*, Computational Statistics and Data Analysis **50**, 7, 1700–1734.

Harm Derksen
 Department of Mathematics
 University of Michigan
 530 Church Street
 Ann Arbor, MI 48109-1043, USA
 hderksen@umich.edu